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THE CONTACT BETWEEN TWO PLATES, ONE OF WHICH CONTAINS A CRACK†

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The problem of the contact between two plates, one of which has a vertical crack which reaches the outer edge, is considered. It is assumed that, in the natural state, the plates are a specified distance from one another. The displacements of points on the plates satisfy two constraints of the inequality type. One of these describes the condition of non-penetration between the plates and is specified at internal points of the region, while the other describes the mutual non-penetration of the edges of the crack and is specified on the boundary of the region. The presence of a crack means that, first, the solution of the problem is sought in a region with a non-smooth boundary, and, second, the boundary conditions on the boundary of the region are given in the form of inequalities. It is proved that the equilibrium problem is solvable. Additional smoothness of the solution up to internal points of the crack is established. It is shown that the problem of controlling external loads with an objective functional, characterizing the opening of the crack, is solvable. For cracks of zero opening it is shown that the solution belongs to class C^∞ in the region of the boundary for smooth external data. The convergence of the solutions of optimal-control problems as the thickness of the plates approaches zero is analysed.

1. FORMULATION OF THE PROBLEM. THE EXISTENCE OF A SOLUTION

Suppose $\Omega \subset R^2$ is a bounded region with an infinitely differentiable boundary Γ, and Γ_ψ is the graph of the function $y = \psi(x), x \in [0, 1], (x, y) \in \bar{\Omega}, \psi \in H_0^3(0, 1)$. We will assume that Γ_ψ and Γ have a single common point—the origin of coordinates (0, 0), and the angle between Γ, and Γ_ψ at the point (0, 0) is positive (Fig. 1). Here $\Omega_\psi \equiv \Omega\backslash\Gamma_\psi$ corresponds to the middle surface of the plate while Γ_ψ is the trace of the crack in the x, y plane. The crack, like the surface in R^3, is described by the relations $y = \psi(x)$, $-\varepsilon \leqslant x \leqslant \varepsilon$, where 2ε is the thickness of the plate. The middle surface of the plate lies in the $z = 0$ plane, and the z axis is directed orthogonal to the x, y plane. The second plate (which has no cracks) can be in contact with the first (containing a crack) and is also $z\varepsilon$ thick.

We will assume that, in the natural state, the plates are situated at a specified distance $\delta \geqslant 0$ from one another (δ = const) and may be in contact with one another in view of the presence of external loads (Fig. 2). The middle surface of the second plate occupies the region Ω. The direction of the z axis is chosen so that the middle surface of the second plate has a negative coordinate z. The first plate will therefore be called the upper plate, and the second will be called the lower plate.

We will denote by $\chi = (W, w)$, $\xi = (U, u)$ the displacement vectors of points of the middle surface of the upper and lower plates respectively, where $W = (w^1, w^2)$, w are the horizontal and vertical displacements of the upper plate, and $U = (u^1, u^2)$ and u are the horizontal and vertical displacements of the lower plate. Suppose $\varepsilon_{ij}(W)$ are the components of the strain tensor of the middle plane of the upper plate while $\sigma_{ij} = \sigma_{ij}(W)$ are the components of the stress tensors in dimensionless form, where

$$\sigma_{11} = \varepsilon_{11} + k\varepsilon_{22}, \quad \sigma_{22} = \varepsilon_{22} + k\varepsilon_{11}, \quad \sigma_{12} = (1-k)\varepsilon_{12}, \quad k = \text{const}, \quad 0 < k < \frac{1}{2} \tag{1.1}$$

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w^i}{\partial x_j} + \frac{\partial w^j}{\partial x_i}\right), \quad i, j = 1, 2, \quad x_1 = x, \quad x_2 = y$$

The energy functional of the upper plate can be written in the form

$$\Pi_f(\chi) = \frac{1}{2}B_\psi(w, w) + \frac{1}{2}\langle\sigma_{ij}(W), \ \varepsilon_{ij}(W)\rangle_\psi - \langle f, \chi\rangle_\psi, \quad f = (f_1, f_2, f_3) \in L^2(\Omega_\psi)$$

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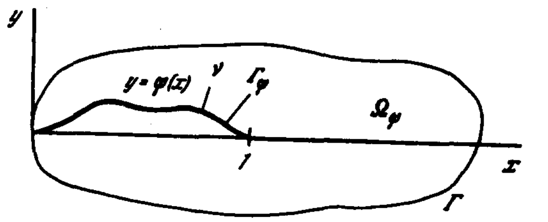

Fig. 1.

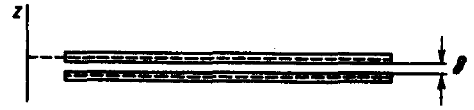

Fig. 2.

$$B_\psi(w,\bar{w}) = \int\limits_{\Omega_\psi} (w_{xx}\bar{w}_{xx} + w_{yy}\bar{w}_{yy} + kw_{xx}\bar{w}_{yy} + kw_{yy}\bar{w}_{xx} + 2(1-k)w_{xy}\bar{w}_{xy})d\Omega_\psi$$

where the brackets $\langle\cdot,\cdot\rangle_\psi$ denote integration over Ω_ψ, f is the vector of the external loads while B_ψ is a bilinear form, characterizing the flexural properties of the plate. Similarly, the energy functional of the lower plate has the form

$$\Pi_g(\xi) = \frac{1}{2}B(u,u) + \frac{1}{2}\langle\sigma_{ij}(U),\ \varepsilon_{ij}(U)\rangle - \langle g,\xi\rangle,\quad g = (g_1,g_2,g_3) \in L^2(\Omega)$$

$$B(u,\bar{u}) = \int\limits_{\Omega} \Delta u \Delta \bar{u}\, d\Omega$$

where the brackets $\langle\cdot,\cdot\rangle$ denote integration over Ω.

The functional of the combined energy of the two plates can therefore be represented in the form $\Pi_f(\chi) + \Pi_g(\xi)$.

As has already been noted, the upper plate has a vertical crack, the form of which is specified. The condition for mutual non-penetration of the crack edges has the form [1]

$$[W]\nu \geqslant \varepsilon|[\partial w/\partial \nu]| \quad \text{on} \quad \Gamma_\psi,\quad \nu = (\nu_1,\nu_2) = (-\psi_x, 1)/\sqrt{1+\psi_x^2} \tag{1.2}$$

where ν is the normal to the graph of Γ_ψ, $[V] = V^+ - V^-$ is the jump in the function V at the crack edges, and the plus and minus superscripts correspond to positive and negative directions of the normal ν respectively.

The plates may interact with one another but the displacement vectors must be such that there is no mutual penetration of points of the plates. The corresponding non-penetration condition can be written in the form

$$w \geqslant u - \delta \ \text{in}\ \Omega_\psi \tag{1.3}$$

We will assume that all the main physical parameters of the lower plate are identical with the parameters of the upper plate. In particular, the relation between the stress and strain tensors for the lower plate area the same as in (1.1).

We will specify the following boundary conditions on the outer boundary

$$w = \partial w/\partial n = W = 0,\ u = \partial u/\partial n = U = 0 \ \text{on}\ \Gamma \tag{1.4}$$

(n is the outward normal to Γ).

We will now formulate the variational form of the problem on the equilibrium of two plates.

Suppose $H^{1,0}(\Omega_\psi)$ is a space of Sobolev functions, having derivatives up to the first order inclusive in Ω_ψ, summed with a square and equal to zero on Γ and $H^{1,0}(\Omega_\psi) \subset H^1(\Omega_\psi)$. Similarly the elements of $H^{2,0}(\Omega_\psi)$ vanish together with the first derivatives on Γ and have derivatives up to the second order inclusive, summed with a square, and $H^{2,0}(\Omega_\psi) \subset H^2(\Omega_\psi)$. We will denote by $H_0^2(\Omega)$ the closure in the norm of $H^2(\Omega)$ of the set of all smooth functions, finite in Ω. Suppose also that $H(\Omega_\psi) = H^{1,0}(\Omega_\psi) \times H^{1,0}(\Omega_\psi) \times H^{2,0}(\Omega_\psi)$, $H_0(\Omega) = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^2(\Omega)$. In a number of places, for brevity, we will use the notation $H = H(\Omega_\psi) \times H_0(\Omega)$.

We will further introduce the following notation for the set of permissible displacements of the plates: $K_\varepsilon = \{(\chi,\xi) \in H(\Omega_\psi) \times H_0(\Omega) |\ (\chi,\xi)$ satisfy conditions (1.2) and (1.3).

The solution of the problem of the equilibrium of two plates, formulated below, will be an element of the set K_ε. In particular, the smoothness of the solution will follow from the fact that it belongs to the space H. The problem of equilibrium can be formulated as a problem of minimizing the functional of the total energy in the set of permissible displacements, namely

$$\inf_{(\chi,\xi)\in K_\varepsilon}\{\Pi_f(\chi)+\Pi_g(\xi)\}$$

In view of the convexity and differentiability of the functional $\Pi_f(\chi) + \Pi_g(\xi)$ in the space H, this problem is equivalent to the variational inequality

$$\Pi'_f(\chi)(\bar{\chi}-\chi)+\Pi'_g(\xi)(\bar{\xi}-\xi)\geq 0,\quad (\chi,\xi)\in K_\varepsilon,\quad \forall(\bar{\chi},\bar{\xi})\in K_\varepsilon \tag{1.5}$$

where $\Pi'_f(\chi)$ and $\Pi'_g(\xi)$ are derivatives of the functions Π_f and Π_g, respectively, at the points χ, ξ. Note, first of all, the correctness of the following inequalities with constants that do not depend on the functions w, u, W and U

$$B_\psi(w,w)\geq c\|w\|^2_{2,\Omega_\psi},\quad \forall w\in H^{2,0}(\Omega_\psi),\quad B(u,u)\geq c\|u\|^2_{2,\Omega},\quad \forall u\in H^2_0(\Omega) \tag{1.6}$$

$$\langle\sigma_{ij}(W),\ \varepsilon_{ij}(W)\rangle_\psi\geq c\|W\|^2_{1,\Omega_\psi},\quad \forall W=(w^1,w^2)\in H^{1,0}(\Omega_\psi) \tag{1.7}$$

where $\|\cdot\|_{s,\Omega_\psi}$ is the norm in $H^{s,0}(\Omega_\psi)$. In addition

$$\langle\sigma_{ij}(U),\ \varepsilon_{ij}(U)\rangle\geq c\|U\|^2_{1,\Omega},\quad \forall U=(u^1,u^2)\in H^1_0(\Omega) \tag{1.8}$$

Inequalities (1.6) are obtained after double use of the Poincaré inequality, while (1.7) and (1.8) are obtained by using the Korn inequality for the regions Ω_ψ and Ω respectively. We introduce the following notation for the bilinear form

$$a(\eta,\bar{\eta})=B_\psi(w,\bar{w})+B(u,\bar{u})+\langle\sigma_{ij}(W),\ \varepsilon_{ij}(\bar{W})\rangle_\psi+\langle\sigma_{ij}(U),\ \varepsilon_{ij}(\bar{U})\rangle \tag{1.9}$$

where $\eta = (\chi, \xi)$, $\chi = (W, w)$, $\xi = (U, u)$, and similarly for $\bar{\eta} = (\bar{\chi}, \bar{\xi})$. By virtue of inequalities (1.6)–(1.8) the following limit holds

$$a(\eta,\eta)\geq c\|\eta\|^2_H,\quad \forall\eta\in H \tag{1.10}$$

Inequality (1.5) can be written in terms of the function $\eta = (\chi, \xi)$ in the form

$$a(\eta,\bar{\eta}-\eta)\geq\langle f,\bar{\chi}-\chi\rangle_\psi+\langle g,\bar{\xi}-\xi\rangle,\quad \forall\bar{\eta}=(\bar{\chi},\bar{\xi})\in K_\varepsilon \tag{1.11}$$

According to inequality (1.10), the functional $\Pi_f(\chi) + \Pi_g(\chi)$ is coercive in H, and, since it is weakly semicontinuous from below, problem (1.5) (or problem (1.11)) has a solution. The solution will be unique. Note also that in the regions Ω_ψ and Ω the following equations are satisfied in the sense of distributions

$$-\sigma_{ij,j}(W)=f_i,\quad -\sigma_{ij,j}(U)=g_i,\quad i=1,2 \tag{1.12}$$

To check the correctness of the first equations it is sufficient to take $(\bar{\chi}, \bar{\xi}) = (\chi + \tilde{\chi}, \xi)$ as test functions in (1.5), where $\tilde{\chi} = (\tilde{W}, 0)$, $\tilde{W}\in (C_0^\infty(\Omega_\psi))^2$ and (χ, ξ) is the solution of (1.5). To check the second equations it is sufficient to choose $(\bar{\chi}, \bar{\xi}) = (\chi, \xi + \tilde{\xi})$ as test functions in (1.5), where $\tilde{\xi} = (\tilde{U}, 0)$, $\tilde{U}\in (C_0^\infty(\Omega))^2$.

We will briefly describe the content of this paper. In Section 2 we will establish the additional regularity of the solution up to internal points Γ_4. Roughly speaking, the smoothness of the solution can be increased by unity compared with the variational smoothness, which is determined by the inclusion $(\chi, \xi) \in K_\varepsilon$. This result will be established using finite differences. Section 3 is devoted to an analysis of the problem of optimal control with a quality functional characterizing the opening of the crack. The main result of this section is to prove the infinite differentiability of the solution in the case when the opening of the crack is zero. Finally, in Section 4 we investigate the limit as $\varepsilon \to 0$, corresponding to the transition from the accurate condition of non-penetration (1.2) to the approximate condition, characterized by the value $\varepsilon = 0$ in (1.2). We note here that in fact the accurate formulation of the problem of the equilibrium of the plates should include the parameter ε not only in the form (1.2). In fact, other quantities also depend on ε, for example, the equations of equilibrium (1.12) should contain ε as a factor in front of $\sigma_{ij,j}(W)$, $\sigma_{ij,j}(U)$, the bilinear forms B_ψ, B should contain the factor ε^3, etc. Hence, this passage to the limit does not indicate that the passage to the limit of the thickness is carried out in the problem of the equilibrium of two plates. It means a transition from the accurate condition of non-penetration (1.2) to the approximate condition corresponding to $\varepsilon = 0$. In particular, it follows from this that, in the context of this paper, the thickness of the lower plate is in fact fixed and its equality to 2ε is not fundamental.

The regularity of the solutions of the boundary-value problems for the equations of the theory of elasticity with non-smooth regions (including the biharmonic equation) was investigated previously [1, 3–9]. The smoothness of the solutions for elliptic equations was also investigated in the case when constraints on the solution of the inequality type are specified on the surface of a lesser dimension [10, 11]. The problem, analysed in this paper, contains both a non-smooth boundary and limitations of the inequality type on the boundary. In fact, the constraints on Γ_ψ must be considered as part of the boundary conditions. The search for a complete set of boundary conditions on Γ_ψ in this case is an independent problem. Inequality (1.2) in this case is one of the boundary conditions, specified on the boundary. Other problems relating to the change in the form of the cracks (and consequently, the change in the form of the region in which a solution is sought), can be found in [12–15].

2. THE REGULARITY OF THE SOLUTION

The purpose of this section is to investigate the smoothness of the solution up to internal points Γ_ψ.

Suppose, first of all, that (χ, ξ) is a solution of problem (1.5). A sphere of radius λ with centre at the point x^0 will be denoted by $R_\lambda(x^0)$. The additional smoothness of the solution in the region of the chosen point, belonging to $\Gamma_\psi\backslash\partial\Gamma_\psi$, will be proved with the additional assumption that Γ_4 is a rectilinear section in the region of this point. We have the following assertion.

Theorem 1. Suppose $x^0 \in \Gamma_\psi\backslash\partial\Gamma_\psi$ and $D(x^0)$ is a neighbourhood of the point x^0 such that $\Gamma_\psi \cap D(x^0)$ is a rectilinear section parallel to the x axis. A $\lambda > 0$ then exists such that

$$W, w_x \in H^2(R_\lambda(x^0)\cap\Omega_\psi), \quad U, u_x \in H^2(R_\lambda(x^0))$$

Proof. We choose a smooth function φ such that $\varphi \equiv 1$ in $R_\lambda(x^0)$ and $\varphi \equiv 0$ outside $R_{3\lambda/2}(x^0)$, $0 \leqslant \varphi \leqslant 1$ everywhere and $\partial\varphi/\partial y = 0$ on Γ_ψ. We will assume that $R_{2\lambda}(x^0) \subset D(x^0)$. We put

$$d_{\pm\tau}p(\bar{x}) = \tau^{-1}(p(\bar{x} \pm \tau e) - p(\bar{x})), \quad \Delta_\tau = -d_{-\tau}d_\tau, \quad 0<|\tau|<\lambda/2$$

where e is the unit vector of the x axis. We introduce the vector (χ_τ, ξ_τ), where

$$\chi_\tau = \chi + \frac{\tau^2}{2}\varphi^2\Delta_\tau\chi, \quad \xi_\tau = \xi + \frac{\tau^2}{2}\varphi^2\Delta_\tau\xi$$

and we will show that $(\chi_\tau, \xi_\tau) \in K_\varepsilon$. To do this we will prove that (1.2) and (1.3) hold. For the function $\upsilon = w - u$ the inequality $\upsilon(\bar{x}) \geqslant -\delta$ obviously holds for $\bar{x} \in \Omega_\psi$. We therefore obtain

$$\upsilon_\tau(\bar{x}) = (w_\tau - u_\tau)(\bar{x}) = \upsilon(\bar{x}) + \frac{\tau^2}{2}\varphi^2(\bar{x})\Delta_\tau\upsilon(\bar{x}) = \upsilon(\bar{x})(1-\varphi^2(\bar{x})) +$$
$$+\frac{\varphi^2(\bar{x})}{2}[\upsilon(\bar{x}-\tau e) + \upsilon(\bar{x}+\tau e)] \geqslant -\delta$$

This denotes that there is an inequality of the form (1.3)

$$w_\tau \geqslant u_\tau - \delta \quad \text{in } \Omega_\psi \tag{2.1}$$

It can be shown, in the same way as in [9], that for $\chi_\tau = (W_\tau, w_\tau)$ the following relation holds

$$[W_\tau]\nu \geqslant \varepsilon\left[\!\left[\frac{\partial w_\tau}{\partial \nu}\right]\!\right] \quad \text{on } \Gamma_\psi \cap D(x^0) \tag{2.2}$$

Consequently, taking into account the fact that the function φ is finite, we conclude that inequality (2.2) also holds on Γ_ψ. Inequalities (2.1) and (2.2) show that for the vector (χ_τ, ξ_τ) conditions (1.2) and (1.3) hold, and therefore $(\chi_\tau, \xi_\tau) \in K_\varepsilon$. This indicates that we can substitute the vector (χ_τ, ξ_τ) into (1.11) as a test vector. This leads to the inequality

$$a(\eta, \varphi^2\Delta_\tau\eta) \geqslant \langle f, \varphi^2\Delta_\tau\chi\rangle_\psi + \langle g, \varphi^2\Delta_\tau\xi\rangle \tag{2.3}$$

It is easy to verify that the upper estimate for the difference between the terms $a(\eta)$, $\varphi^2\Delta_\tau\eta)$ and $-a(d_\tau(\varphi\eta), d_\tau(\varphi\eta))$ can be found in terms of the right-hand side of inequality (2.4) written below, so that it follows from (2.3) that

$$a(d_\tau(\varphi\eta), d_\tau(\varphi\eta)) \leqslant c\{\|\eta\|_H^2 + \|d_\tau(\varphi\eta)\|_H(\|\eta\|_H + \|f\|_{0,\Omega_\psi} + \|g\|_{0,\Omega})\} \tag{2.4}$$

with a constant c which independent of τ. Bearing (1.10) in mind, we conclude from (2.4) that

$$\|d_\tau(\varphi\chi)\|_{H(\Omega_\psi)} + \|d_\tau(\varphi\xi)\|_{H_0(\Omega)} \leqslant c \tag{2.5}$$

where the constant c is independent of τ. It follows from (2.5) that

$$\frac{\partial}{\partial x}(\varphi\chi) \in H(\Omega_\psi), \quad \frac{\partial}{\partial x}(\varphi\xi) \in H_0(\Omega)$$

and hence we have the following inclusions

$$\begin{aligned} &W_x \in H^1(R_\lambda(x^0) \cap \Omega_\psi), \quad U_x \in H^1(R_\lambda(x^0)) \\ &w_x \in H^2(R_\lambda(x^0) \cap \Omega_\psi), \quad u_x \in H^2(R_\lambda(x^0)) \end{aligned} \tag{2.6}$$

In the region Ω_ψ, Eqs (1.12) for W can be written in the form

$$W_{yy} = F$$

By virtue of (2.6) the inclusion $F \in L^2(R_\lambda(x^0) \cap \Omega_\psi$ holds, so that $W_{yy} \in L^2(R_\lambda(x^0) \cap \Omega_\psi)$. In addition, by Eqs (1.12) the equations $U_{yy} = G$ hold for U in the neighbourhood of the point x^0, where, in view of (2.6), $G \in L^2(R_\lambda(x^0)$. This proves Theorem 1.

The theorem which follows gives additional smoothness compared with Theorem 1 for the case when there is no contact between the plates in the neighbourhood of a fixed point of the crack.

Theorem 2. Suppose all the conditions of the previous theorem are satisfied, and in addition

$$w^{\pm}(x^0) > u(x^0) - \delta \tag{2.7}$$

Then

$$\begin{aligned} &W \in H^2(R_\lambda(x^0) \cap \Omega_\psi), \quad U \in H^2(R_\lambda(x^0)) \\ &w \in H^3(R_\lambda(x^0) \cap \Omega_\psi), \quad u \in H^3(R_\lambda(x^0)) \end{aligned} \tag{2.8}$$

Proof. We conclude from (2.7) and (1.5) that there is a neighbourhood $D(x^0)$ of the point x^0 such that in $D(x^0) \cap \Omega_\psi$ the following equation holds, in the sense of distributions

$$\Delta^2 w = f_3 \tag{2.9}$$

We will use the following fact, the proof of which can be found in [16]. Suppose $D \subset R^2$ is a bounded region with a fairly smooth boundary and υ is a distribution on D which possesses the property υ, $\nabla\upsilon \in H^{-1}(D)$. Then $\upsilon \in L^2(D)$, and a constant c, which depends on D, exists such that

$$\|\upsilon\|_{L^2(D)} \leqslant c\{\|\upsilon\|_{H^{-1}(D)} + \|\nabla\upsilon\|_{H^{-1}(D)}\}$$

It follows from (2.6) that $\partial(\varphi w)/\partial x \in H^{2,0}(\Omega_\psi)$. Hence, in the neighbourhood of the point x^0 the derivatives of w_{xxx}, w_{yyx}, w_{xxy} belong to L^2. Equation (2.9) in $D(x^0) \cap \Omega_\psi$ can be written in the form

$$w_{yyyy} = h$$

As proved above, the functions h, w_{yyy}, w_{yyyx} belong to $H^{-1}(\Omega_\psi \cap D)$, where D is a certain neighbourhood of the point x^0. Consequently, the functions w_{yyy} belong to $L^2(\Omega_\psi \cap D_1)$ and we have the following limit

$$\|w_{yyy}\|^2_{L^2(\Omega_\psi \cap D_1)} \leqslant c\{\|w_{yyy}\|^2_{H^{-1}(\Omega_\psi \cap D_1)} + \|w_{yyyy}\|^2_{H^{-1}(\Omega_\psi \cap D_1)} + \|w_{yyyx}\|^2_{H^{-1}(\Omega_\psi \cap D_1)}\}$$

where D_1 is a neighbourhood of the point x^0, $\overline{D}_1 \subset D$. Hence, we obtain the required inclusion (2.8) for w. In addition, the following equation holds in $D(x^0)$, in the sense of distributions

$$\Delta^2 u = g_3 \tag{2.10}$$

so that, proceeding in the same way as above, we obtain inclusion (2.8) for the function u also. This proves Theorem 2.

3. THE OPTIMAL CONTROL PROBLEM. CRACKS OF MINIMUM OPENING

In this section we will investigate the problem of the control of external loads (f, g) with a quality functional

$$J_\varepsilon(f,g) = \int_{\Gamma_\psi} |[\chi]| d\Gamma_\psi$$

characterizing the opening of the crack (see [17]). As previously, (χ, ξ) is the solution of boundary-value problem (1.5), corresponding to the right-hand side of (f, g). At the first stage we will prove the theorem on the existence of a solution of the optimal-control problem. We will further show that the solutions corresponding to cracks of zero aperture are infinitely differentiable for infinitely differentiable f and g. Despite the fact that, in this section, the parameter ε will be fixed, the dependence of the quality functional on ε will be pointed out. This is due to the fact that, later in Section 4, we will investigate the limit as $\varepsilon \to 0$.

Suppose $F \times G \subset L^2(\Omega_\psi) \times L^2(\Omega)$ is a convex, closed bounded set, and $(f, g) \in F \times G$. We then have the following assertion.

Theorem 3. A solution of the minimization problem

$$\inf_{F \times G} J_\varepsilon(f,g) \tag{3.1}$$

exists.

Proof. Suppose $(f_n, g_n) \in F \times G$ is a minimizing sequence. For each n we can obtain a unique solution of the problem

$$\Pi'_{f_n}(\chi_n)(\bar{\chi} - \chi_n) + \Pi'_{g_n}(\xi_n)(\bar{\xi} - \xi_n) \geqslant 0, \quad \forall(\bar{\chi}, \bar{\xi}) \in K_\varepsilon \tag{3.2}$$

In view of the boundedness of f_n and g_n in $L^2(\Omega)$, the following limit follows from (3.2)

$$\|\chi_n\|_{H(\Omega_\psi)} + \|\xi_n\|_{H_0(\Omega)} \leqslant c \tag{3.3}$$

which is uniform in n. Choosing, if necessary, a subsequence, we can assume that as $n \to \infty$

$$(\chi_n, \xi_n) \to (\chi, \xi) \text{ weakly in } H, \text{ strongly in } L^2(\Omega);\ [\chi_n] \to [\chi] \text{ strongly in } L^1(\Gamma_\psi)$$

This convergence enables us to take the limit as $n \to \infty$ in (3.2) and we obtain

$$\Pi'_f(\chi)(\bar{\chi}-\chi)+\Pi'_g(\xi)(\bar{\xi}-\xi)\geq 0, \quad (\chi,\xi)\in K_\varepsilon, \quad \forall(\bar{\chi},\bar{\xi})\in K_\varepsilon$$

which means that $\chi = \chi(f, g)$, $\xi = \xi(f, g)$. Hence

$$\inf_{F\times G} J_\varepsilon(\bar{f},\bar{g}) = \lim_{n\to\infty}\inf J_\varepsilon(f_n,g_n)\geq J_\varepsilon(f,g)\geq \inf_{F\times G} J_\varepsilon(\bar{f},\bar{g})$$

and, consequently, the pair (f, g) obtained in fact solves optimal-control problem (3.1). This proves Theorem 3.

It turns out that if the opening of the crack is zero in the neighbourhood of a certain point $x^0 \in \Gamma_\psi$, for which $w^\pm(x^0) > u(x^0) - \delta$, and the right-hand sides of f and g are infinitely differentiable in the neighbourhood of this point, the solution of problem (1.5) is also infinitely differentiable in the region of the point x^0. Our further discussion is devoted to justifying this assertion. The result on the smoothness of the solution will be proved for the case when x^0 is the point of intersection of Γ and Γ_ψ, i.e. $x^0 \in \Gamma \cap \Gamma_\psi$. The case when $x^0 \notin \Gamma \cap \Gamma_\psi$ can be investigated more simply. It is mentioned in the note after the proof of Theorem 4.

First, we note that, in the same way as was described earlier in [9], we can obtain the form of the boundary conditions in the neighbourhood of an arbitrary point $\bar{x} \in \Gamma_\psi\backslash\partial\Gamma_\psi$ by assuming that the solution $\eta = (\chi, \xi)$ is sufficiently smooth, and that the inequality $w^\pm(\bar{x}) > u(\bar{x}) - \delta$ holds. The last inequality denotes that there is no contact between the two plates at the point x. That is, in addition to (1.2) the following boundary conditions hold (for simplicity we will assume that $\varepsilon = 1$)

$$[\sigma_\nu(W)]=0, \quad \sigma_s(W)=0, \quad [m(w)]=0, \quad t(w)=0 \quad \text{on } \Gamma_\psi \tag{3.4}$$

$$|m(w)|\leq -\sigma_\nu(W), \quad m(w)\left[\frac{\partial w}{\partial \nu}\right]+\sigma_\nu(W)[W]\nu=0 \quad \text{on } \Gamma_\psi \tag{3.5}$$

Here $m(w)$ and $t(w)$ are the bending moment and shear force on Γ_ψ, defined by the formulae

$$m(w)=k\Delta w+(1-k)\frac{\partial^2 w}{\partial \nu^2}, \quad t(w)=\frac{\partial}{\partial \nu}\Delta w+(1-k)\frac{\partial^3 w}{\partial \nu \partial^2 s}, \quad s=(-\nu_2,\nu_1)$$

and $\sigma_\nu(W)$ and $\sigma_s(W)$ are the normal and tangential components of the vector of the forces on Γ_ψ

$$\{\sigma_{ij}(W)\nu_j\}=\sigma_\nu(W)\nu+\sigma_s(W)s$$

The above boundary conditions must be understood formally in the sense that they hold on the assumption that the solution $\eta = (\chi, \xi)$ of problem (1.11) is fairly smooth. It is important that (1.2) and (3.4) (3.5) give a complete set of boundary conditions on Γ_ψ in the following sense. If the equations of equilibrium and conditions (1.2), (1.4) and (3.4) (3.5) hold, we can infer variational inequality (1.11). In fact, below only some of the boundary conditions (3.4) and (3.5) will be necessary.

Thus, we will formulate the main assertion of this section, relating to cracks of zero opening, i.e. cracks possessing the property $[\chi] = 0$.

Theorem 4. Suppose $\delta > 0$, $x^0 \in \Gamma \cap \Gamma_\psi$. We will assume that $[\chi] = 0$ on $\Gamma_\psi \cap D(x^0)$ and $f, g \in C^\infty(D(x^0) \cap \bar{\Omega}$, where $D(x^0)$ is a certain neighbourhood of the point x^0. A neighbourhood $D_1(x^0)$ of the point x^0 then exists such that the solution of problem (1.5) possesses the property

$$\chi,\xi\in C^\infty(D_1(x^0)\cap\bar{\Omega})$$

Proof. The open set $D(x^0) \cap \Omega_\psi$ can be represented in the form of the union $D(x^0) \cap \Omega_\psi = D^+ \cup D^-$, where the regions $D^\pm$ correspond to positive and negative directions of the normal ν, i.e. for $\bar{x} \in D^\pm$ the inequalities $y > \psi(x), y < \psi(x), \bar{x} \equiv (x, y)$, respectively, hold. In view of the assumption that the angle between Γ and Γ_ψ at the point x^0 is larger than zero, we can use the imbedding theorem, according to which the functions w and u will be continuous in $\bar{\Omega} = \Omega \cup \Gamma$ and $\bar{\Omega}_\psi = \Omega_\psi \cup \Gamma \cup \Gamma_\psi^\pm$, respectively. Hence, the inequality $\delta > 0$ ensures that the relation $w > u - \delta$ holds in a certain neighbourhood $D(x^0)$ of the point x^0. In particular $w^{\pm}(\bar{x}) > u(\bar{u}) - \delta, \bar{x} \in D(x^0) \cap \Gamma_\psi$. So in D^+ and D^- the following equation holds, in the sense of distributions

$$\Delta^2 w - f_3 = 0$$

This equation also holds in $D(x^0) \cap \Omega$.

In fact, $[\chi] = 0$ on $\Gamma_\psi \cap D(x^0)$, so that from (1.2) we obtain $[\partial w/\partial \nu] = 0$ on $\Gamma_\psi \cap D(x^0)$. This means that $w \in H^2(D(x^0) \cap \Omega)$ (see [18]). Bearing in mind the boundary conditions on $\Gamma_\psi \cap D(x^0)$ of the form $t(w) = 0$, $[m(w)] = 0$, as previously [1] it can be shown that

$$(\Delta^2 w - f_3, \varphi) = 0, \quad \forall \varphi \in C_0^\infty(D(x^0) \cap \Omega) \tag{3.6}$$

which also proves the above assertion. The brackets $(\cdot, \varphi)$ in (3.6) denote the action of the distribution on the element φ.

Similarly, the condition $[\chi] = 0$ on $\Gamma_\psi \cap D(x^0)$ ensures the inclusion $W \in H^1(D(x^0) \cap \Omega)$. Hence, taking into account the boundary conditions $\sigma_{ij}\nu_j = 0$ on $\Gamma_\psi \cap D(x^0)$ $(t = 1, 2)$, in the same way as in [1], it can be shown that

$$(\sigma_{ij,j}(W) + f_i \varphi) = 0, \quad \forall \varphi \in C_0^\infty(D(x^0) \cap \Omega), \quad i = 1, 2$$

The inequality $w^{\pm}(\bar{x}) > u(\bar{x}) - \delta, \bar{x} \in D(x^0) \cap \Gamma_\psi$ also ensures that Eq. (2.10) holds in $D(x^0) \cap \Omega$. This indicates that the following equations hold in $D(x^0) \cap \Omega$

$$\Delta^2 w = f_3, \quad \Delta^2 u = g_3, \quad -\sigma_{ij,j}(W) = f_i, \quad -\sigma_{ij,j}(U) = g_i, \quad i = 1, 2$$

Since the right-hand sides f_i and g_i here are functions that are infinitely differentiable in $D(x^0) \cap \bar{\Omega}$, we obtain the statement of the theorem (see [19, 20]).

Note. If $x^0 \in \Gamma_\psi, x^0 \notin \Gamma \cap \Gamma_\psi$ and $w^{\pm}(x^0) > u(x^0) - \delta$, the equality $|\chi| = 0$ on $\Gamma_\psi \cap D(x^0)$ also ensures that the solution χ, ξ is infinitely differentiable in the neighbourhood of $D(x^0)$, provided that $f, g \in C^\infty(D(x^0))$. In other words, in this case the inclusion $\chi, \xi \in C^\infty(D(x^0))$ holds.

This assertion can be proved in the same way as Theorem 4, if we note that the inequality $w(\bar{x}) > u(\bar{x}) - \delta$, holds for all $\bar{x} \in D_1(x^0) \cap \Gamma_\psi$, where $D_1(x^0)$ is a neighbourhood of the point x^0. Moreover $w^{\pm}(\bar{x}) > u(\bar{x}) - \delta$, $\bar{x} \in D_1(x^0) \cap \Gamma_\psi$.

4. THE LIMIT AS $\varepsilon \to 0$

Consider the case of the approximate description of the condition of non-penetration of the crack edges, corresponding formally to $\varepsilon = 0$ in (1.2). As in the case when $\varepsilon > 0$, when $\varepsilon = 0$ we can consider the problem of the equilibrium of two plates and prove the existence of a solution of the optimal-control problem with a quality functional characterizing the opening of the crack edges. The purpose of this section is to investigate the convergence of the solutions of the optimal-control problems of the form (3.1) as $\varepsilon \to 0$. We will assume that Γ_ψ is a rectilinear section parallel to the x axis.

Thus, the limiting conditions of non-penetration, obtained from (1.2) and (1.3) have the form

$$[W]\nu \geqslant 0 \text{ on } \Gamma_\psi, \quad w \geqslant u - \delta \text{ in } \Omega_\psi \tag{4.1}$$

We will introduce a set of permissible displacements of the plates, corresponding to constraints (4.1)

$$K_0 = \{(\chi, \xi) \in H(\Omega_\psi) \times H_0(\Omega) | (\chi, \xi) \text{ satisfy conditions } (4.1)\} \tag{4.2}$$

Suppose the set $F \times G$ is chosen in the same way as before. For each fixed element $(f, g) \in F \times G$ we can obtain a unique solution of the variational inequality

$$\Pi'_f(\chi)(\bar{\chi} - \chi) + \Pi'_g(\xi)(\bar{\xi} - \xi) \geqslant 0, \quad (\chi, \xi) \in K_0, \quad \forall (\bar{\chi}, \bar{\xi}) \in K_0 \tag{4.3}$$

We again consider the quality functional describing the opening of the crack

$$J_0(f,g)=\int_{\Gamma_\psi}|[\chi]|d\Gamma_\psi$$

In this case the function χ corresponds to the external loads (f, g) and is found from (4.3).

A unique solution of the problem of the optimal control of external loads

$$\inf_{F\times G} J_0(f,g) \tag{4.4}$$

exists. The proof of this fact is simpler than the proof of Theorem 3, and we will therefore not dwell on it.

Suppose $(\chi_\varepsilon, \xi_\varepsilon, f_\varepsilon, g_\varepsilon)$ corresponds to the solution of optimal-control problem (3.1) for a given ε, i.e. $(f_\varepsilon, g_\varepsilon)$ is the solution of problem (3.1) while $(\chi_\varepsilon, \xi_\varepsilon)$ is obtained from (1.5) when $(f, g) = (f_\varepsilon, g_\varepsilon)$. The aim of the discussion below is to prove the assertion relating the solutions of optimal-control problems (3.1) and (4.4). We have the following result.

Theorem 5. We can choose from the sequence $(\chi_\varepsilon, \xi_\varepsilon, f_\varepsilon, g_\varepsilon)$ a subsequence, denoted as previously, such that

$$(\chi_\varepsilon, \xi_\varepsilon) \to (\chi, \xi) \text{ weakly in } H(\Omega_\psi)\times H_0(\Omega)$$

$$f_\varepsilon, g_\varepsilon \to f, g \text{ weakly in } L^2(\Omega),\ m_\varepsilon \to m_0$$

$$m_\varepsilon = \inf_{F\times G} J_\varepsilon(f,g),\quad m_0=\inf_{F\times G} J_0(f,g)$$

Here (χ, ξ, f, g) corresponds to the solution of optimal-control problem (4.4).

Proof. Suppose $\chi_\varepsilon(f, g)$, $\xi_\varepsilon(f, g)$ is the solution of variational inequality (1.5) for given fixed f and g. We will take $(\bar\chi, \bar\xi) \in K_{\varepsilon 0}$. Then $(\bar\chi, \bar\xi) \in K_\varepsilon$ for all $\varepsilon \leqslant \varepsilon_0$. We substitute $(\bar\chi, \bar\xi)$ into inequality (1.5) as a test function. We obtain

$$\|\chi_\varepsilon(f,g)\|_{H(\Omega_\psi)}+\|\xi_\varepsilon(f,g)\|_{H_0(\Omega)}\leqslant c \tag{4.5}$$

uniformly with respect to $\varepsilon \leqslant \varepsilon_0$. Choosing, if necessary, a subsequence we can assume that as $\varepsilon \to 0$

$$\chi_\varepsilon(f,g)\to\tilde\chi \text{ weakly in } H(\Omega_\psi),\quad \xi_\varepsilon(f,g)\to\tilde\xi \text{ weakly in } H_0(\Omega) \tag{4.6}$$

$$\tilde\chi_\varepsilon^\pm(f,g)\to\tilde\chi^\pm \text{ strongly in } L^1(\Gamma_\psi) \tag{4.7}$$

$$\varepsilon\left[\!\left[\frac{\partial w_\varepsilon(f,g)}{\partial \nu}\right]\!\right]\to 0 \text{ strongly in } L^2(\Gamma_\psi) \tag{4.8}$$

We will take an arbitrary fixed element $(\bar\chi, \bar\xi) \in K_0$ and construct, using the lemma proved above, the sequence $(\bar\chi_\varepsilon, \bar\xi_\varepsilon) \in K_\varepsilon$, which converges strongly in $H(\Omega_\psi) \times H_0(\Omega)$ to $(\bar\chi, \bar\xi)$. We then substitute the elements of this sequence as test functions into the inequality

$$\Pi'_f(\chi_\varepsilon)(\bar\chi-\chi_\varepsilon)+\Pi'_g(\xi_\varepsilon)(\bar\xi-\xi_\varepsilon)\geqslant 0,\ (\chi_\varepsilon,\xi_\varepsilon)\in K_\varepsilon,\ \forall(\bar\chi,\bar\xi)\in K_\varepsilon$$

Using (4.6) we can here take the limit as $\varepsilon \to 0$. Condition (4.8) ensures that the inclusion $(\tilde\chi, \tilde\xi) \in K_0$ holds. The limit variational inequality will then have the form

$$\Pi'_f(\tilde\chi)(\bar\chi-\tilde\chi)+\Pi'_g(\tilde\xi)(\bar\xi-\tilde\xi)\geqslant 0,\ (\tilde\chi,\tilde\xi)\in K_0,\ \forall(\bar\chi,\bar\xi)\in K_0$$

which denotes that $\tilde\chi = \chi(f, g)$, $\tilde\xi = \xi(f, g)$. Consequently, we then obtain from (4.7)

$$J_\varepsilon(f,g)\to J_0(f,g),\ \varepsilon\to 0 \tag{4.9}$$

Suppose now that (f, g) is the solution of optimal-control problem (4.4), (4.3). By (4.9) we obtain $m_\varepsilon \leqslant J_\varepsilon(f, g) \to J_0(f, g) = m_0$, and therefore

$$\limsup m_\varepsilon \leqslant m_0 \tag{4.10}$$

On the other hand, from the fact that the set $F \times G$ is bounded in the space $L^2(\Omega_\psi) \times L^2(\Omega)$ we have

$$\| (f_\varepsilon, g_\varepsilon) \|_{L^2(\Omega)} \leqslant c \tag{4.11}$$

uniformly with respect to ε. Consequently, from the variational inequalities

$$\Pi'_{f_\varepsilon}(\chi_\varepsilon)(\bar{\chi} - \chi_\varepsilon) + \Pi'_{g_\varepsilon}(\xi_\varepsilon)(\bar{\xi} - \xi_\varepsilon) \geqslant 0, \quad (\chi_\varepsilon, \xi_\varepsilon) \in K_\varepsilon, \quad \forall (\bar{\chi}, \bar{\xi}) \in K_\varepsilon \tag{4.12}$$

we derive the following limit, uniform with respect to ε

$$\| \chi_\varepsilon \|_{H(\Omega_\psi)} + \| \xi_\varepsilon \|_{H_0(\Omega)} \leqslant c \tag{4.13}$$

We can assume from (4.11) and (4.13) that as $\varepsilon \to 0$

$$f_\varepsilon, g_\varepsilon \to f, g \text{ weakly in } L^2(\Omega) \tag{4.14}$$

$$(\chi_\varepsilon, \xi_\varepsilon) \to (\chi_0, \xi_0) \text{ weakly in } H, \quad \text{strongly in } L^2(\Omega) \tag{4.15}$$

$$\varepsilon \left| \left[\frac{\partial w_\varepsilon}{\partial \nu} \right] \right| \to 0 \quad \text{strongly in } L^2(\Gamma_\psi) \tag{4.16}$$

$$\chi_\varepsilon^\pm(f, g) \to \tilde{\chi}_0^\pm \text{ strongly in } L^1(\Gamma_\psi) \tag{4.17}$$

Again bearing the lemma in mind, using (4.14)–(4.17) we take the limit as $\varepsilon \to 0$ in (4.12). We finally obtain the variational inequality

$$\Pi'_{f_0}(\chi_0)(\bar{\chi} - \chi_0) + \Pi'_{g_0}(\xi_0)(\bar{\xi} - \xi_0) \geqslant 0, \quad (\chi_0, \xi_0) \in K_0, \quad \forall (\bar{\chi}, \bar{\xi}) \in K_0$$

which indicates that $\chi_0 = \chi(f_0, g_0)$, $\xi_0 = \xi(f_0, \xi_0)$.

As before, it can be shown that $J_\varepsilon(f_\varepsilon, g_\varepsilon) \to J_0(f_0, g_0)$ as $\varepsilon \to 0$, and

$$\liminf m_\varepsilon \geqslant J_0(f_0, g_0) \tag{4.18}$$

From (4.10) and (4.18) we obtain that (f_0, g_0) is the solution of optimal-control problem (4.4), (4.3) and $m_\varepsilon \to m_0$. This proves Theorem 5.

We will now justify the auxiliary assertion used to prove Theorem 5. We recall that Γ_ψ is assumed here to be a rectilinear section parallel to the x axis.

Lemma. For any fixed element $(\bar{\chi}, \bar{\xi}) \in K_0$, a sequence $(\bar{\chi}_\varepsilon, \bar{\xi}_\varepsilon) \in K_\varepsilon$ exists such that $(\bar{\chi}_\varepsilon, \bar{\xi}_\varepsilon) \to (\bar{\chi}, \bar{\xi})$ strongly in $H(\Omega_\psi) \times H_0(\Omega)$.

Proof. We extend the graph of Γ_ψ outside $x = 1$ smoothly so that the extension intersects the boundary Γ at a non-zero angle (Fig. 3). The region Ω_ψ is then divided into two regions: Ω_1, Ω_2 with Lipschitz boundaries $\partial\Omega_1$, $\partial\Omega_2$. As everywhere previously, the boundaries Γ_ψ^+ and Γ_ψ^- are assumed to be different. The fact that the function $(\bar{\chi}, \bar{\xi})$ belongs to the set K_0 means that the following inequalities hold

$$[\bar{W}]\nu \geqslant 0 \quad \text{on } \Gamma_\psi, \quad \bar{w} \geqslant \bar{u} - \delta \quad \text{in } \Omega_\psi$$

and the fact that the functions $(\bar{\chi}_\varepsilon, \bar{\xi}_\varepsilon)$ belong to the set K_ε denotes that the following inequalities hold

$$[\bar{W}_\varepsilon]\nu \geqslant \varepsilon |[\partial \bar{w}_\varepsilon / \partial \nu]| \quad \text{on } \Gamma_\psi, \quad \bar{w}_\varepsilon \geqslant \bar{u}_\varepsilon - \delta \quad \text{in } \Omega_\psi$$

To prove the lemma it is sufficient to construct the sequence $(\bar{\chi}_\varepsilon, \bar{\xi}_\varepsilon)$ of the form $(\bar{\chi}_\varepsilon, \bar{\xi})$ such that $(\bar{\chi}_\varepsilon, \bar{\xi}_\varepsilon) \in K_\varepsilon$ and

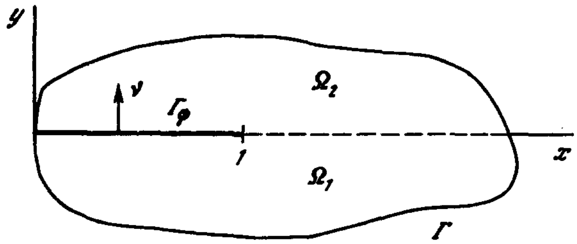

Fig. 3.

$$(\tilde{\chi}_\varepsilon, \tilde{\chi}) \text{ strongly in } H(\Omega_\psi) \tag{4.19}$$

Note that if the function $\tilde{W} \in [H^{1,0}(\Omega_\psi)]^2$, which has the property

$$[\tilde{W}]\nu = |[\partial\bar{w}/\partial\nu]| \quad \text{on} \quad \Gamma_\psi \tag{4.20}$$

is constructed, while the functions $\tilde{\chi}_\varepsilon = (\bar{W}_\varepsilon, \bar{w}_\varepsilon)$ are defined in the region Ω_ψ by the formula $(\bar{W}_\varepsilon, \bar{w}_\varepsilon) = (\bar{W} + \varepsilon\tilde{W}, \bar{w})$, the sequence $(\tilde{\chi}_\varepsilon, \xi)$ will be the required sequence. In fact, the convergence of (4.19) is obvious and

$$[\bar{W}_\varepsilon]\nu \geqslant \varepsilon|[\partial\bar{w}_\varepsilon/\partial\nu]| \quad \text{on} \quad \Gamma_\psi, \quad \bar{w}_\varepsilon \geqslant \bar{u} - \delta \quad \text{in} \quad \Omega_\psi$$

To construct the function $\tilde{W}$, possessing property (4.20), we note that $\upsilon = (0, 1)$ on Γ_4. Since $\bar{w} \in H^2(\Omega_\psi)$, we have $\bar{w}_y|_{\Omega_i} \in H^1(\Omega_i)$ $(i = 1, 2)$, and hence the inclusions $\bar{w}_y|_{\partial\Omega_i} \in H^{1/2}(\partial\Omega_i)$ $(i = 1, 2)$ hold (see [21]).

Consider the following function on $\partial\Omega_1$

$$h_\gamma(\bar{x}) = \begin{cases} \min\{-[\bar{w}_y(\bar{x})], [\bar{w}_y(\bar{x})]\}, & \bar{x} \in \Gamma_\psi \\ 0, & \bar{x} \notin \Gamma_\psi \end{cases}$$

Then $h_\gamma \in H^{1/2}(\partial\Omega_1)$. Suppose $h \in H^1(\Omega_1)$ is the extension of the function $h\gamma$ into the region Ω_1. Note that if we extend h by zero into Ω_2 we obtain a function in Ω_ψ which belongs to $H^1(\Omega_\psi)$. This extension, as before, will be denoted by h. We can now define the vector function $\tilde{W}$ as follows: $\tilde{W} = (0, h)$ in Ω_ψ. In this case

$$[\tilde{W}]\nu = \max\{-[\bar{w}_y], [\bar{w}_y]\} = |[\bar{w}_y]| \quad \text{on} \quad \Gamma_\psi$$

where $|[\bar{w}_y]| = |[\partial\bar{w}/\partial\upsilon]|$ on Γ_ψ.

Thus, we have constructed a function $\tilde{W} \in [H^{1,0}(\Omega_\psi)]^2$ having the required property (4.20), which completes the proof of the lemma.

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